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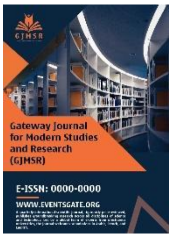

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Endo Second Submodules, Endo Quasi-Prime Second Submodules, and Related Concepts

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Abstract: The notions of endo second submodule and endo quasi-prime second submodule are presented as generalizations of second submodule and quasi-prime second sub mod, where the submodule is not zero. Endo second submodule is the name given to B of V as an A-module, if $\forall f \in End(V)$, then either $f(\mathcal{B}) = \mathcal{B}$ or $f(\mathcal{B}) = 0$, and a non-zero submodule $\mathcal{B}$ of V as A-module is called an endo quasi-prime second submodule, if $f, g \in End(V)$ and S is completely irreducible of V. then $\mathcal{B}$ is endo quasi-prime second submodule either $f(\mathcal{B}) \subseteq S$ or $g(\mathcal{B}) \subseteq S$. Let $\mathcal{B}$ be a sub mod of an A-module V that is not zero. We state that $\mathcal{B}$ is TAB second sub mod of V, if whenever $a, b \in T$, L is a completely irreducible submodule of V, and $\mathcal{B}ab \subseteq L$, then $\mathcal{B}a \subseteq L$ or $\mathcal{B}b \subseteq L$or $a \cdot b \in Ann_T(\mathcal{B})$. This might be thought of as a twin idea of the TAB sub mod. These notions are defined by the action of the endomorphism ring on modules, offering a more profound structural viewpoint that connects the endomorphism behavior of internal module attributes with a variety of characteristics, as well as certain fundamental characteristics and theorems of these concepts. Additionally, it shows how these ideas connect to various types of second submodules.

Keywords: Second Modules, Endo Essential Second Modules, Endo Small Second Modules, Endo Large Maximal Submodules.

1. Introduction:

In this article, each ring is connected to identity, and each module is a unitary right module. In (Yassemi, 2001) proposed the concept of the second module, which is defined as an A-module V if $V \neq 0$ and for each $t \in A$, either $Vt = (0)$ or $Vt = V$. Equivalently V is second module if for each ideal H of A, either $VH = (0)$ or $VH = V$. (Kasch,1982) introduced the idea of an essential submod. in which A-module V's submodule $\mathcal{B}$ is referred to as essential (large) submodule of V and represented by $\mathcal{B} \leq_e V$ if whenever $\mathcal{B} \cap L = (0), L \leq V$, then $L = (0)$. (Wisbauer,1991) introduced the concept of small submodule. of an A-module V where $\mathcal{B}$ is a valid submodule, is known as small submodule of V and denoted by $\mathcal{B} \ll V$ if $\mathcal{B} + L \neq V$ for any appropriate submodule L of V. (Hadi, Shyaa & Al-aeashi, 2019) developed the concept of the crucial second module, in which an A-module V is an endo essentially second (merely endo ess second) if for each $f \in End(V)$, either $f(V) = (0)$ or $f(V) \leq_e V$. (Hadi, I.M.A., Hameed, & Abbas, M.K.,2024). introduced the idea of an endo small second submodule when an A-module V's submodule B is named endo small second submodule if for each $f \in End(V)$, either $f(\mathcal{B}) = \mathcal{B}$ or $f(\mathcal{B}) \ll \mathcal{B}$. (Al-Chalabi, H.A., Hanoon, W.H., Allami, M.W., & Hameed, Z.N. , 2026) presented the concept of large maximal second module, The A-module V is called endo large maximal second submodule (merely endo LM second) If an ideal H of A for a proper, then either $f(V) = (0)$ or $f(V) <_{LM} V$ where a proper submodule $\mathcal{B}$ of V is a large maximal (simply LM) submodule if there exists a submodule L of V $\mathcal{B} < L \leq V$, then L is essential submodule of V . (Payrovi, Sh., & Babaei, S. , 2012) . A non-zero submodule B of an A-module V is called a generalized 2-absorbing second submodule whenever $a, b \in T$, L is a completely irreducible submod of V and $\mathcal{B}ab \subseteq L$, then $\mathcal{B}a \in L$ or $\mathcal{B}b \in L$ or $a \cdot b \in Ann_T(\mathcal{B})$. A module that is a generalized 2-absorbing second submodule of itself is called a generalized 2-absorbing second module. (Faith, C. , 1973) and (Faith, C. ,1973). An submodule $\mathcal{B}$ of V is completely irreducible, if $\mathcal{B} = \cap_{i \in I} B_i$, where $\{B_i\}_{i \in I}$ is a family of submodules of V, implies $\mathcal{B} = B_i$ for some i ∈ I (Fuchs, L., Heinzer, W., & Olberding, B. , 2005). A proper submodule $\mathcal{B}$ of a A - mod V is referred to as visible if there is a non-zero suitable ideal C of A such that $\mathcal{B}$ = C $\mathcal{B}$ (Fiadh, M.S., Shihab, B.N., & Abed, M.H. , 2024). (Anderson, F.W., & Fuller, K.R., 1992) referred to as the submodule $\mathcal{B}$ of a left A-module V is called pure if VI∩ $\mathcal{B}$ = $\mathcal{B}$ I for every ideal $I$of the ring A (Darani, A.Y., & Soheilnia, F. , 2011) and (Shyaa, F.D. , 2004).

The primary purpose of this research is to examine second and quasi-prime second submodules under the influence of endomorphism, where the endo second includes second and maximal conditions needs. The endo second submodule of V is a first concept non-zero submodule $\mathcal{B}$ of V as an A-module , if ∀ $f \in End(V)$, then either $f(\mathcal{B}) = \mathcal{B}$ or $f(\mathcal{B}) = 0$.

The second idea nonzero submodule $\mathcal{B}$ of V as A-module is called an endo quasi-prime second submodule, if $f, g \in End(V)$ and S is completely irreducible of V. then $\mathcal{B}$ is endo quasi Prime Second submodule (simply EQP Second) if whenever $f \circ g(\mathcal{B}) \subseteq$S implies either $f(\mathcal{B}) \subseteq S$ or $g(\mathcal{B}) \subseteq S$.

This study is split into two sections. The endo second submodules and endo quasi-prime second submodules, as well as their properties and connections, are introduced in the first portion.

2. Endo Second Sub mod.s

In this segment, we present the notion of the endo second submodule and discuss its features and examples.

2.1 Definition

Give $\mathcal{B} \neq 0$ be a submodule of A-module of V $\mathcal{B}$ is known as the endo second submodule of V, if ∀ $f \in End(V)$then either $f(\mathcal{B}) = P$ or $f(\mathcal{B}) = 0$.

An A-module V is endo second module if it is itself an endo second submodule

2.2 Examples

1) Consider Z_6as Z-module then $(\bar{2})$,$(\bar{3})$ are Endo Second submodule of Z_6 where $f \in End(Z_6)$ such that $f(\bar{x}) = 3\bar{x}$, $\forall \bar{x} \in (\bar{2})$, $f((\bar{2})) = 0$ and $f((\bar{3})) = (\bar{3})$.

2) Consider Z_{12} as Z-module, then $(\bar{3})$ is not Endo Second submodule of Z_{12} where $f(\bar{x}) = 2\bar{x}$ therefore $f((\bar{3})) = (\bar{6}) \neq (\bar{3})$ and $f((\bar{3})) \neq 0$.

2.3 Proposition

Let $\mathcal{B} \neq 0$ be a submodule of A-module V and A is commutative. If $\mathcal{B}$ is endo second submodule, then $\mathcal{B}$ is second submodule.

Proof:

Suppose $f \in End(V)$ such that $f(x) = r.x$,$\forall x \in V$,$r \in T$ and $\mathcal{B} \neq 0$ is a submodule of V ,but $\mathcal{B}$ is endo second submodule, so that $f(\mathcal{B}) = \mathcal{B}$ or $f(\mathcal{B}) = 0$, then $r.\mathcal{B} = \mathcal{B}$ or $r.\mathcal{B} = 0$. Thus, $\mathcal{B}$i s second submodule.

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2.4 Remarks and Examples

1. 2.3 Proposition's opposite is generally false, for instance: see as Z-module, and, then is second submodule of Z2Z2 since or , but is not endo second submodule of since and$Z_2 \oplus Z_2 f(\bar{x}, \bar{y}) = (\bar{x}, \bar{0})$, $\forall(\bar{x}, \bar{y}) \in Z_2 \oplus Z_2$ $f \in End(Z_2 \oplus Z_2)(\bar{1}, \bar{1}) \oplus r \cdot (\bar{1}, \bar{1}) = (\bar{1}, \bar{1})$ $(\bar{0}, \bar{0}), \forall r \in Z(\bar{1}, \bar{1}) Z_2 \oplus Z_2 f((\bar{1}, \bar{1})) = (\bar{1}, \bar{0}) \neq (\bar{1}, \bar{1})$ $f(\bar{1}, \bar{1}) \neq (\bar{0}, \bar{0})$.
2. Every endo second submodule is an end LM second submodule, whereas the converse is generally false. For instance, if Z is a Z-module , when $f(x) = 2x$, $\forall x \in 9Z$ then 9Z is an endo LM second submodule of Z since $f(9Z) = \{18,36,54,72,90\} = 18Z$, when 18Z is LM second submodule since $18Z \subseteq \{2Z, 3Z, 6Z, 18Z\}$ and $2Z, 3Z, 6Z, 18Z$ are essential submodule of Z. But $f(9Z) \neq 9Z$ and $f(9Z) \neq 0$Therefore, $9Z$ is not endo second submodule of Z.
3. Every endo second submodule is an end ess second submodule, In general, the inverse is untrue; for instance: Let Z_{24} as Z-module and let $(\bar{2})$ is end ess second submodule if $f(\bar{x}) = 2\bar{x}$, $\forall \bar{x} \in (\bar{2})$,$f \in End(Z_{24})$, then $f(\bar{2}) = (\bar{4}) \leq_e (\bar{2})$, but $(\bar{2})$ is not endo second submodule Since $f(\bar{2}) \neq (\bar{2})$ and $f(\bar{2}) \neq 0$.
4. All endo second submodule of V is endo small second submodule of V; however, the reverse is generally erroneous, for example: Let Z_{24} as Z-module and $(\bar{3})$ is endo small second submodule if $f(\bar{x}) = 2\bar{x}, \forall \bar{x} \in (\bar{3})$,$f \in End(Z_{24})$, so that $f((\bar{3})) = (\bar{6}) \ll (\bar{3})$, but $(\bar{3})$ is not endo second submodule since $f(\bar{3}) \neq (\bar{3})$ and $f(\bar{3}) \neq 0$.

2.5 Remark

Assume that two sub mods of an A-module V are N and K. If L is a totally irreducible submodule of V such that K ⊆ L, then N ⊆ L. This is sufficient to prove N ⊆ K. (Ansari-Toroghy, H., & Farshadifar, F. , 2011) .

2.6 Proposition

If $P \neq 0$ is fully invariant a submodule of A-module V, then the following statements are equivalent:

1. $\mathcal{B}$ is endo second submodule of V.
2. If $f \in End(V)$and $f(\mathcal{B}) \subseteq K$ where K is a submodule of V then either $f(\mathcal{B}) = 0$ or $\mathcal{B} \subseteq K$.
3. If $f \in End(V)$and $f(\mathcal{B}) \subseteq K$ where K is completely irreducible submodule of V then $f(\mathcal{B}) = 0$ or $\mathcal{B} \subseteq K$.

Proof: (1)⇒ (2) Let $\mathcal{B} \neq 0, f(\mathcal{B}) \subseteq K$, $f \in End(V)$ where K be a submodule of V, but $\mathcal{B}$ is endo second submodule, hence $f(\mathcal{B}) = \mathcal{B}$ or $f(\mathcal{B}) = 0$ since $f(\mathcal{B})) \subseteq K$, so that $\mathcal{B} \subseteq K$ or $f(\mathcal{B}) = 0$.

(2) ⇒ (3) It is clear

(3) ⇒ (1) Let $f \in End(V)$and $f(\mathcal{B})) \subseteq K$ where K is completely irreducible submodule of V, so that from (3), we have $f(\mathcal{B}) = 0$ or $\mathcal{B} \subseteq K$ and $f(\mathcal{B}) \subseteq K$ by assume, then $f(\mathcal{B}) \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq f(\mathcal{B})$ from Remark 2.5 hence $f(\mathcal{B}) = 0$ or $f(\mathcal{B}) = \mathcal{B}$, therefor $\mathcal{B}$ is endo second submodule of V.

2.7 Proposition

If R is a pure submodule of a scalar endo second A-module V, then R is an endo second submodule

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Proof:
Since R is pure submodule , then $RI = VI \cap R$ and since V is endo second module, then V is second module by Proposition 2.3, therefore $VI = V$ or $VI = 0$, if $VI = V$, then $RI = V \cap R = R$, now if $VI = 0$, then $RI = (0) \cap R = (0)$. Thus, R is end second submodule

2.8 Proposition
Every endo second submodule is visible submodule

Proof:
Let $\mathcal{B} \neq 0$ be a submodule of A-module and suppose $f \in End(V)$, such that $f(x) = r.x$, $\forall x \in V$, $r \in T$ and $\mathcal{B} \neq 0$ is a submodule of V ,but $\mathcal{B}$ is Endo Second submodule, so that $f(\mathcal{B}) = \mathcal{B}$ or $f(\mathcal{B}) = 0$, then $r.\mathcal{B} = \mathcal{B}$. Thus, $\mathcal{B}$ is visible submodule.

2.9 Remark:
The reverse of Proposition (2.8) wrong (see the example in Remarks and Examples (2.4) part (1)).

3. Endo Quasi-Prime Second Sub mods

This section explains the Endo Quasi-Prime second submodule notion, discusses its features, and provides examples.

3.1 Definition
Let $P \neq 0$ be a submodule of A-mod V , $f, g \in End(V)$ and S a completely irreducible of V. Then $\mathcal{B}$ is endo quasi-prime second submodule (simply EQP second) if whenever $f \circ g(\mathcal{B}) \subseteq S$ implies either $f(\mathcal{B}) \subseteq S$ or $g(\mathcal{B}) \subseteq S$.

An A-module V is EQP second module if it is itself an EQP second submodule.

3.2 Example
Think of Z_{12} as Z-module and $f, g \in End(Z_{12})$ so as to $f(\bar{x}) = 2\bar{x}$, $g(\bar{x}) = \bar{x}$, $\forall \bar{x} \in (\bar{3})$ where $(\bar{2})$ is completely irreducible and $(\bar{3})$ is EQP Second submodule since $f \circ g\big((\bar{3})\big) = (\bar{6}) \subseteq (\bar{2})$, then $f\big((\bar{3})\big) = (\bar{6}) \subseteq (\bar{2})$.

3.3 Proposition
Let $\mathcal{B} \neq 0$ be a submodule of A-module V and $\mathcal{B}$ is endo second submodule, then $\mathcal{B}$ is EQP second submodule.

Proof:
Let $f, g \in End(V)$ and S is completely irreducible so as to $f \circ g(\mathcal{B}) \subseteq S$, so that $f\big(g(\mathcal{B})\big) = f(\mathcal{B}) = \mathcal{B}$or $f\big(g(\mathcal{B})\big) = f(0) = 0$, since $\mathcal{B}$ is Endo second. Hence $f\big(g(\mathcal{B})\big) = \mathcal{B} \subseteq S$, then $f(\mathcal{B}) \subseteq S$ and $g(\mathcal{B}) \subseteq S$. Thus, $\mathcal{B}$ is EQP second submodule.

3.4 Remark
Conversely of Proposition 3.3 is incorrect usually, for example: consider Z_{24} a Z-module, and $f, g \in End(Z_{24})$ so as to $f(\bar{x}) = 2\bar{x}$, $g(\bar{x}) = 3\bar{x}$, $\forall \bar{x} \in Z_{24}$ where $(\bar{2})$ is completely irreducible then $f \circ g\big((\bar{3})\big) = f\big(g(\bar{3})\big) = f\big((\bar{3})\big) = (\bar{6}) \subseteq (\bar{2})$. Hence $(\bar{3})$ is EQP second

submodule, but $(\bar{3})$ is not endo second submodule since $f\big((\bar{3})\big) = (\bar{6}) \neq (\bar{3})$ and $f\big((\bar{3})\big) = (\bar{6}) \neq (\bar{0})$.

3.5 Remarks

Let $\mathcal{B} \neq 0, S \neq 0$ be two submodules of A-module V. The following claims are then held.:

1. If $\mathcal{B} \subseteq S$. $\mathcal{B}$ is EQP second submodule of V, then $\mathcal{B}$ is EQP second submodule of S.

2. If $\mathcal{B}$ is EQP second submodule of V, then $\mathcal{B} \cap S$ is EQP second submodule of $\mathcal{B} + S$.

Proof (1)

Let $f, g \in End(V)$ and B is completely irreducible of A-module V, since S is submodule of V such that $f \circ g(\mathcal{B}) \subseteq B$, since $\mathcal{B} \subseteq S$, but $\mathcal{B}$ is EQP second submodule , so that either $f(\mathcal{B}) \subseteq B \; or \; g(\mathcal{B}) \subseteq B$. Thus, $\mathcal{B}$ is EQP second submodule of S.

Proof (2)

Let $f, g \in End(V)$ and $\mathcal{B}$,S are two EQP second submodules, but $\mathcal{B} \cap S \subseteq S$, so that by Remark 3.5 part (1), we get $\mathcal{B} \cap S$ is EQP second submodule of $\mathcal{B} + S$.

Let $\mathcal{B}$ be a non-zero submodule of an A-module V. We say that $\mathcal{B}$ is a 2-absorbing second submodule of V if whenever $a, b \in A$, L is a completely irreducible submodule of V, and $\mathcal{B}ab \subseteq L$, then $\mathcal{B}a \subseteq L$ or $\mathcal{B}b \subseteq L$or $a \cdot b \in Ann_A(\mathcal{B})$. This is the dual notion of the 2-absorbing submodule.(Ansari-Toroghy, H., & Farshadifar, F. , 2019)

3.6 Proposition

Let $\mathcal{B} \neq 0$ be a submodule of A-module V , if $\mathcal{B}$ is EQP second submodule of V, then $\mathcal{B}$ is TAB second submodule of A-module V.

Proof

Let $f, g \in End(V)$, $f(x) = xa$, $g(x) = xb$, $\forall a, b \in A$,where $a \cdot b \notin AnnP$ and B is completely irreducible of A-module V such that $f \circ g(\mathcal{B}) = \mathcal{B}ab \subseteq B$, but $\mathcal{B}$ is EQP second submodule , then we have $f(\mathcal{B}) = \mathcal{B}a \subseteq B$ or $g(\mathcal{B}) = \mathcal{B}b \subseteq B$. Thus, $\mathcal{B}$ is TAB second submodule in V.

3.7 Remark

Conversely of Proposition 3.6 incorrect generally, as an example : consider $Z_3 \oplus Z_3$ as Z_3-module , $\langle(\bar{1},\bar{2})\rangle$ is completely irreducible where $f(\bar{x},\bar{y}) = (\bar{x},\bar{0})$ and $g(\bar{x},\bar{y}) = (\bar{0},\bar{y})$ $\forall f, g \in End\ (Z_3 \oplus Z_3)$, then $\langle(\bar{1},\bar{2})\rangle$={$(\bar{0},\bar{0})$, $(\bar{1},\bar{2})$,$(\bar{2},\bar{1})$} so $\langle(\bar{1},\bar{2})\rangle$ is TAB second submodule in ($Z_3 \oplus Z_3$) since $2 \cdot 0 \cdot (\bar{1},\bar{2}) \in \langle(\bar{1},\bar{2})\rangle$ where $2{,}0 \in Z_3$ and $(\bar{2},\bar{1}) \in (Z_3 \oplus Z_3)$ then $2 \cdot (\bar{1},\bar{2}) \in \langle(\bar{1},\bar{2})\rangle$ and $0 \cdot (\bar{1},\bar{2}) \in \langle(\bar{1},\bar{2})\rangle$ and $2 \cdot 0 \in$ann$(\bar{2},\bar{1})$, but $\langle(\bar{1},\bar{2})\rangle$ is not EQP second submodule in $(Z_3 \oplus Z_3)$ since $f \circ g\langle(\bar{1},\bar{2})\rangle = (\bar{0},\bar{0}) \in \langle(\bar{1},\bar{2})\rangle$ then $f(\langle(\bar{1},\bar{2})\rangle) = (\bar{1},\bar{0}) \notin \langle(\bar{1},\bar{2})\rangle$ and $g(\langle(\bar{1},\bar{2})\rangle) = (\bar{0},\bar{2}) \notin (\bar{1},\bar{2})$, therefor $\langle(\bar{1},\bar{2})\rangle$ is not EQP second submodule in $(Z_3 \oplus Z_3$).

3.8 Corollary (Yaseen, S.M., & Hasan, W.Kh. , 2012)

A-module V is a scalar module for all finitely generated multiplications.

3.9 Proposition

Let $\mathcal{B} \neq 0$ be a submodule of a scalar A-module V and A is commutative , $\mathcal{B}$ is EQP second submodule of V if and only if $\mathcal{B}$ is TAB second submodule of V.

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Proof

⇒)By Proposition 3.6

⇐) Let $f, g \in End(V)$, $f(x) = xa$, $g(x) = xb$ $\forall a, b \notin AnnP$, a, b $\in A$, $\forall x \in V$ *and* B is completely irreducible of A-module V , such that $f \circ g(\mathcal{B}) \subseteq B$, but V is scalar module , So that $f \circ g(\mathcal{B}) = \mathcal{B}ab \subseteq B$, but $\mathcal{B}$ is TAB second submodule of , V then we have $Pa \subseteq B$ or $\mathcal{B}b \subseteq B$. Hence $f(\mathcal{B}) \subseteq B$ or $g(\mathcal{B}) \subseteq B$. Thus, $\mathcal{B}$ is EQP second submodule of V.

3.10 Proposition

Let $\mathcal{B} \neq 0$ be a submodule of finitely generated multiplication A-module V , $\mathcal{B}$ is EQP second submodule of V if and only if $\mathcal{B}$ is TAB second submodule of V.

Proof

From Corollary 3.8 and Proposition 3.9, the proposition is proved.

3.11 Proposition

Let $\mathcal{B} \neq 0$ be a submodule of a duo A-module V and B is completely irreducible of A-module V , then $\mathcal{B}$ is EQP second submodule of V if and only if $(B:_V \mathcal{B}) = (B:_V f(\mathcal{B}))$.where $f(\mathcal{B}) \nsubseteq B$, $f \in End(V)$.

Proof

⇒)Let $g \in (B:_V \mathcal{B})$, Hence $g(\mathcal{B}) \subseteq B$, But V is a duo A-module , So $f(\mathcal{B}) \subseteq \mathcal{B}$, it follows that $g(f(\mathcal{B})) \subseteq B$,So we have $g \in (B:_V f(\mathcal{B}))$. Now, Let $g \in (B:_V f(\mathcal{B}))$.and $f(\mathcal{B}) \nsubseteq B$, So we have $g(f(\mathcal{B})) \subseteq B$, But $\mathcal{B}$ is EQP second (SEQP second) submodule of V , Then $g(\mathcal{B}) \subseteq B$, Hence $g \in (B:_V \mathcal{B})$. Thus $(B:_V \mathcal{B}) = (B:_V f(\mathcal{B}))$.

⇐) Let $f, g \in End(V)$ and B is completely irreducible (B is submodule) of A-module V such that $f \circ g(\mathcal{B}) \subseteq B$ and $f(\mathcal{B}) \nsubseteq B$, So $g \in (B:_V f(\mathcal{B}))$., But $(B:_V \mathcal{B}) = (B:_V f(\mathcal{B}))$. , So that $(B:_V \mathcal{B})$. Hence $g(\mathcal{B}) \subseteq B$. Thus, $\mathcal{B}$ is EQP second submodule of V.

3.12 Proposition.

If $\mathcal{B} \neq 0$ is fully invariant submodule of A-module V, then statements are comparable :

a) $\mathcal{B}$ is EQP second submodule of V;

b) $f \circ g(\mathcal{B}) = f(\mathcal{B})$ or $f \circ g(\mathcal{B}) = g(\mathcal{B})$.

c) $f \circ g(\mathcal{B}) \subseteq L_1 \cap L_2$ where L_1, L_2 are completely irreducible submodules of A, implies either that $f(\mathcal{B}) \subseteq L_1 \cap L_2$ or $g(\mathcal{B}) \subseteq L_1 \cap L_2$.

Proof

(a) ⇒ (b) Assume that $f \circ g(\mathcal{B}) \subseteq L$, where L is completely irreducible submodule of V, by part (a) , implies either $f(\mathcal{B}) \subseteq L$ or $g(\mathcal{B}) \subseteq L$, then by Remark 2.5, we have $f(\mathcal{B}) \subseteq f \circ g(\mathcal{B})$ or $g(\mathcal{B}) \subseteq f \circ g(\mathcal{B})$, but $\mathcal{B}$ is fully invariant submodule, so that $f \circ g(\mathcal{B}) \subseteq f(\mathcal{B})$ or $f \circ g(\mathcal{B}) \subseteq g(\mathcal{B})$. Thus, $f \circ g(\mathcal{B}) = f(\mathcal{B})$ or $f \circ g(\mathcal{B}) = g(\mathcal{B})$.

(b) ⇒ (a) Let B be completely irreducible submodule with $f \circ g(\mathcal{B}) \subseteq$ B. By part (b), we have $f \circ g(\mathcal{B}) = f(\mathcal{B})$ or $f \circ g(\mathcal{B}) = g(\mathcal{B})$. Thus either $f(\mathcal{B}) = f \circ g(\mathcal{B}) \subseteq$ B or $g(\mathcal{B}) = f \circ g(\mathcal{B}) \subseteq$ B. Thus, $\mathcal{B}$ is EQP second submodule of V

(a) ⇒ (c) Assume that $f \circ g(\mathcal{B}) \subseteq L_1 \cap L_2$, where L_1, L_2 are completely irreducible submodules of A, hence $f \circ g(\mathcal{B}) \subseteq L_1 \cap L_2 \subseteq L_1$ and $f \circ g(\mathcal{B}) \subseteq L_1 \cap L_2 \subseteq L_2$, by part (c), implies either that $f(\mathcal{B}) \subseteq L_1$ or $g(\mathcal{B}) \subseteq L_1$ also $f(\mathcal{B}) \subseteq L_2$ or $g(\mathcal{B}) \subseteq L_2$, then that $f(\mathcal{B}) \subseteq L_1 \cap L_2$ or $g(\mathcal{B}) \subseteq L_1 \cap L_2$.

(c) ⇒ (b) similarly by (a) ⇒ (b), we get the result.

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3.13 Lemma. Let ′ be a monomorphism of A-modules. If L′ is a completely irreducible submodule of , thenis a completely irreducible submodule of M. (Ansari-Toroghy, H., & Farshadifar, F. , 2019)$f: M \rightarrow Mf(M)\ f^{-1}\ (L')$

3.14 Lemma. Let be a monomorphism of A-moduleules. If L is a completely irreducible sub mod of M, then is a completely irreducible submodule of (Ansari-Toroghy, H., & Farshadifar, F. , 2019)$f : M \rightarrow {}'Mf(L)f(M)$.

3.15 Proposition

Let $f : V \rightarrow V'$ be an epimorphism of A-module, where End(V) is commutative ring. Then we have the following statements:

1) If $\mathcal{B}$ is fully invariant and EQP second submodule of V such that $\ker f \subseteq \mathcal{B}$, then $f(\mathcal{B})$ is EQP second submodule of V'.

2) If $\acute{\mathcal{B}}$ is EQP second submodule of V′ and $\mathcal{B}' \subseteq f(V)$, then $f^{-1}(\mathcal{B}')$ is EQP second submodule of V.

Proof:

(1) If $f(\mathcal{B}) = 0$,then $ff^{-1}(\mathcal{B}) = 0$,so that $\mathcal{B} = 0$ that is contradiction,

Then $f(\mathcal{B}) \neq 0$,let g, h $\in End(V)$,and K is completely irreducible submodule of V ,such that $g \circ$ h$(f(P) \subseteq$ K,then $g \circ$ h$(f^{-1}f(\mathcal{B})) \subseteq f^{-1}$(K), Hence $g \circ$ h$(\mathcal{B}) \subseteq f^{-1}$(K), Since $\mathcal{B}$ is EQP second submodule of V and f^{-1}(K) is completely irreducible submodule of V, by [Lemma 3.13] , either $g(\mathcal{B}) \subseteq f^{-1}$(K) or h$(\mathcal{B}) \subseteq f^{-1}$(K), then $g(f(\mathcal{B})) \subseteq ff^{-1}(K)$ or $h(f(\mathcal{B})) \subseteq ff^{-1}(K)$,since $End(V)$ is commutative ring,

$$g(f(\mathcal{B})) \subseteq (\mathrm{K}) \text{ or } h(f(\mathcal{B})) \subseteq (K)$$

Therefore $f(\mathcal{B})$ is EQP second submodule of V' as needed.

(2) If $f^{-1}(\acute{\mathcal{B}}) = 0$,then $f(V) \cap \mathcal{B} = ff^{-1}(\mathcal{B}') = f(0) = 0$,so that $\mathcal{B}' = 0$ that is contradiction,

Then $f^{-1}(\mathcal{B}') \neq 0$,let g, h $\in End(V)$,and B is completely irreducible submodule of V ,such that $g \circ$ h$(f^{-1}(\mathcal{B}')) \subseteq$ B,then $g \circ$ h$\left(ff^{-1}(\mathcal{B}')\right) \subseteq f(B)$, Hence $g \circ$ h$(\mathcal{B}') \subseteq f(B)$, Since $\acute{\mathcal{B}}$ EQP second submodule of V′ and $f(B)$ is completely irreducible submodule of V′, by [Lemma 3.14] , either $g(\mathcal{B}') \subseteq f(B)$ or h$(\mathcal{B}') \subseteq f(B)$, $g(f^{-1}(\mathcal{B}')) \subseteq ff^{-1}(B)$ or h$(f^{-1}(\mathcal{B}')) \subseteq ff^{-1}(B)$,since $End(V)$ is commutative ring,

$$g(f^{-1}(\mathcal{B}')) \subseteq (\mathrm{B}) \text{ or } \mathrm{h}f^{-1}(\mathcal{B}')) \subseteq (\mathrm{B})$$

Therefore, $f^{-1}(\mathcal{B}')$ is EQP second submodule of V.

3.16 Proposition

Let $\mathcal{B} \neq 0$ be a submodule of A-module V. If $\mathcal{B}$ is EQP second submodule of V , then S^{-1} $\mathcal{B}$ is EQP second submodule of S^{-1}V.

Proof

Let $\frac{f}{s_1}, \frac{g}{s_2} \in End(S^{-1}\mathrm{V})$, such that $\frac{f \circ g}{s_1 s_2}(S^{-1}\mathcal{B}) \subseteq S^{-1}(K)$ where K is completely irreducible submodule then there exists $r \in S$, such that $r(f \circ g)(\mathcal{B}) \subseteq K$, so that $f \circ g(r\mathcal{B}) \subseteq K$, but $\mathcal{B}$ is EQP second submodule of V, then $f(r\mathcal{B}) \subseteq K$ or $g(r\mathcal{B}) \subseteq K$, so that $\frac{rf(\mathcal{B})}{rs_1s_3} = \frac{f(S^{-1}\mathcal{B})}{s_1s_3} \subseteq S^{-1}K$ or $\frac{rg(\mathcal{B})}{rs_2s_3} = \frac{g(S^{-1}\mathcal{B})}{s_2s_3} \subseteq S^{-1}K$. Thus, S^{-1} $\mathcal{B}$ is EQP second submodule of S^{-1}V.

4. Results and Discussion

1. Every endo second submodule is second submodule and visible submodule, but the converse is not true in general; for example,, see Remarks and Examples 2.4 part (1).

2. Every endo second submodule is an ELM second submodule whereas the converse is generally false. For instance: see Remarks and Examples 2.4 part (2).

3. Every endo second submodule is endo small second submodule, but the converse is not true in general; for example,, see Remarks and Examples 2.4 part (4).

4. Every pure submodule of a scalar endo second module is endo second submodule.

5. Every endo second submodule is EQP second submodule, but the converse is not true in general; for example,, see Remark 3.4 .

6. Every EQP second submodule is TAB second submodule, but the converse is true under the scalar-module condition in proposition 3.10.

7. If $\mathcal{B}$ is EQP second submodule of V, then $S^{-1}\mathcal{B}$ is EQP second submodule of $S^{-1}V$.

5. Conclusion

In conclusion, the main objective of this research is to analyze second and quasi-prime second submodules under the influence of endomorphism., every endo second submodule is second submodule, but the converse is not true. In general, every endo second submodule is an ELM second submodule, end second submodule, and endo small second submodule, and visible submodule whereas the converse is generally false. Also, every pure submodule of a scalar endo second module is endo second submodule. Furthermore, every endo second submodule is EQP second submodule, but the converse is not true in general. Every EQP second submodule is TAB second submodule, but the converse is true under the scalar-module condition in proposition 3.10, also the homomorphic image and inverse homomorphic image of EQP second submodule is EQP second submodule. Finally, if $\mathcal{B}$ is EQP second submodule of V, then $S^{-1}\mathcal{B}$ is EQP second submodule of $S^{-1}V$.

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